MINIMIZATION OF FREQUENCY-WEIGHTED l_2-SENSITIVITY FOR MULTI-INPUT/MULTI-OUTPUT LINEAR SYSTEMS

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The problem of minimizing a frequency-weighted l_2-sensitivity measure subject to l_2-scaling constraints is considered for multi-input/multi-output (MIMO) linear discrete-time systems. The constrained optimization problem is converted into an unconstrained optimization problem by using linear-algebraic techniques. An efficient quasi-Newton algorithm with closed-form formula for gradient evaluation is then applied to solve the unconstrained optimization problem. Finally, the optimal system structure is constructed by employing the resulting coordinate transformation matrix that minimizes the frequency-weighted l_2-sensitivity measure subject to the scaling constraints. A numerical example is also presented to illustrate the utility of the proposed technique.

1. INTRODUCTION

The synthesis of a multi-input/multi-output (MIMO) linear discrete-time system with a given transfer function matrix is an important research topic, since the state-space equations corresponding to the transfer function matrix are not unique. Naturally, among the infinite number of realizations of the transfer function matrix, it is often desirable to identify a state-space realization that minimizes a suitable sensitivity measure. When realizing a fixed-point state-space description with finite word length (FWL) from a prescribed transfer function matrix with infinite accuracy coefficients, the coefficients in the state-space description must be truncated or rounded to fit the FWL constraints. This coefficient quantization usually alters the characteristics of the system. For instance, a stable system may be turned to an unstable one. This motivates the study of the coefficient sensitivity minimization problem. In [1–12], two main classes of techniques have been explored for constructing state-space descriptions

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that minimize the coefficient sensitivity: l_1/l_2- sensitivity minimization [1–6] and l_2- sensitivity minimization [7–12]. It has been argued in [7–12] that the sensitivity measure based on the l_2 norm is more natural and reasonable relative to that based on the l_1/l_2- sensitivity minimization. Alternatively, it is well known that the use of scaling constraints can be beneficial for suppressing overflow oscillations [13,14]. Either l_2- sensitivity minimization problem or frequency-weighted l_2- sensitivity minimization problem subject to l_2 – scaling constraints for SISO (single-input/single-output) state-space digital filters has been solved iteratively by converting it into an unconstrained optimization problem with an appropriate linear transformation [15, 16]. However, to our best knowledge, there is no study on the minimization of frequency-weighted l_2 – sensitivity subject to l_2 – scaling constraints for MIMO linear discrete-time systems.

In this paper, the problem of minimizing a frequency-weighted l_2- sensitivity measure subject to l_2- scaling constraints for MIMO linear discrete-time systems is investigated. First, an expression for evaluating the frequency-weighted l_2 – sensitivity is introduced and the frequency-weighted l_2- sensitivity minimization problem subject to l_2- scaling constraints is formulated. Next, the constrained optimization problem is converted into an unconstrained one by using linear algebraic techniques. An efficient quasi-Newton algorithm [17] is then applied to solve the unconstrained optimization problem. A numerical example is also presented to illustrate the utility of the proposed technique.

2. PROBLEM FORMULATION

Consider a stable, controllable and observable MIMO linear discrete-time system $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})_n$ described by:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}x(k)+\boldsymbol{B}u(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}x(k)+\boldsymbol{D}u(k) \end{aligned}, \tag{1}$$

where $\boldsymbol{x}(k)$ is an $n\times 1$ state-variable vector, $u(k)$ is a $q\times 1$ input vector, $\boldsymbol{y}(k)$ is a $p\times 1$ output vector, and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are real constant matrices of appropriate dimensions. The transfer function of the linear system in (1) is given by:

$$\boldsymbol{H}(z)=\boldsymbol{C}(z\boldsymbol{I}_n-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}, \tag{2}$$

whose $(i,j)^{\text{th}}$ element is described by

$$H_{ij}(z)=\boldsymbol{c}_i(z\boldsymbol{I}_n-\boldsymbol{A})^{-1}\boldsymbol{b}_j+d_{ij}, \tag{3}$$

where

$$\boldsymbol{B} = [b_1 \quad b_2 \quad \cdots \quad b_q],$$

$$\boldsymbol{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1q} \\ d_{21} & d_{22} & \cdots & d_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pq} \end{bmatrix}. \tag{4}$$

The frequency-weighted l_2- sensitivity of the linear system in (1) is defined as follows.

Definition 1. Let $\boldsymbol{X}$ be an $m \times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is then defined as:

$$\boldsymbol{S}_X = \frac{\partial f}{\partial X}, \quad (\boldsymbol{S}_X)_{ij} = \frac{\partial f}{\partial x_{ij}}. \tag{5}$$

Definition 2. Let $\boldsymbol{X}(z)$ be an $m \times n$ complex matrix-valued function of a complex variable z, and let $x_{pq}(z)$ be the (p, q)th entry of $\boldsymbol{X}(z)$. The l_2- norm of $\boldsymbol{X}(z)$ is then defined as:

$$\left\|\boldsymbol{X}(z)\right\|_2 = \left(\mathrm{tr}\left[\frac{1}{2\pi j} \oint_{|z|=1} \boldsymbol{X}(z)\boldsymbol{X}^{*}(z)\frac{\mathrm{d}z}{z} \right] \right)^{\frac{1}{2}}. \tag{6}$$

From (3) and Definitions 1 and 2, the overall frequency-weighted l_2-sensitivity measure for the linear system in (1) is defined as:

$$\begin{aligned} S &= \sum_{i=1}^{p}\sum_{j=1}^{q}\left\| W_A(z)\frac{\partial H_{ij}(z)}{\partial \boldsymbol{A}} \right\|_2^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}\left\| W_B(z)\frac{\partial H_{ij}(z)}{\partial \boldsymbol{b}_j} \right\|_2^2 + \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{q}\left\| W_C(z)\frac{\partial H_{ij}(z)}{\partial c_i^T} \right\|_2^2, \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}\left\| W_A(z)\left[\boldsymbol{f}_j(z)\boldsymbol{g}_i(z)\right]^{\mathrm{T}} \right\|_2^2 + \\ &\quad + q\sum_{i=1}^{p}\left\| W_B(z)\boldsymbol{g}_i^{\mathrm{T}}(z) \right\|_2^2 + p\sum_{j=1}^{q}\left\| W_c(z)\boldsymbol{f}_j(z) \right\|_2^2, \end{aligned} \tag{7}$$

where $\boldsymbol{f}_j(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}_j$ and $\boldsymbol{g}_i(z) = \boldsymbol{c}_i(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}$. Since term $\boldsymbol{D}$ in (2) and the sensitivities with respect to its elements are independent of the state-space coordinate, they are neglected in (7). The frequency-weighted l_2-sensitivity measure in (7) can be expressed as:

$$S = \sum_{i=1}^{p}\sum_{j=1}^{q} \mathrm{tr}\left[\boldsymbol{M}_A(\boldsymbol{I}_n)_{ij}\right] + q\,\mathrm{tr}\left[\boldsymbol{W}_B\right] + p\,\mathrm{tr}\left[\boldsymbol{K}_c\right] \tag{8}$$

where $\boldsymbol{M}_A(\boldsymbol{I}_n)_{ij}$, $\boldsymbol{W}_B$, and $\boldsymbol{W}_C$ are obtained by the following general expression:

$$\boldsymbol{X} = \frac{1}{2\pi j}\oint_{|z|=1} \boldsymbol{Y}(z)\boldsymbol{Y}^*(z)\frac{\mathrm{d}z}{z} \tag{9}$$

with

$$\begin{aligned}
\boldsymbol{Y}(z) &= W_A(z)\left[\boldsymbol{f}_j(z)\boldsymbol{g}_i(z)\right]^{\mathrm{T}} \ \text{ for } \ \boldsymbol{X} = \boldsymbol{M}_A(\boldsymbol{I}_n)_{ij} \\
\boldsymbol{Y}(z) &= W_B^*(z)\left[\boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\right]^* \ \text{ for } \ \boldsymbol{X} = \boldsymbol{W}_B \\
\boldsymbol{Y}(z) &= W_c(z)(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} \ \text{ for } \ \boldsymbol{X} = \boldsymbol{K}_c.
\end{aligned} \tag{10}$$

The matrices $\boldsymbol{K}_C$, W_B, and $\boldsymbol{M}_A(\boldsymbol{I}_n)_{ij}$ can be computed using:

$$\begin{aligned}
\boldsymbol{K}_c &= \sum_{l=0}^{\infty} \boldsymbol{F}_c(l)\boldsymbol{F}_C^{\mathrm{T}}(l)\ , \ \boldsymbol{W}_B = \sum_{l=0}^{\infty} \boldsymbol{G}_B^{\mathrm{T}}(l)\boldsymbol{G}_B(l), \\
\boldsymbol{M}_A(\boldsymbol{I}_n)_{ij} &= \sum_{l=0}^{\infty} \boldsymbol{H}_A^{\mathrm{T}}(l)_{ij}\boldsymbol{H}_A(l)_{ij},
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\boldsymbol{F}_c(l) &= \sum_{k=0}^{l} w_C(k)\boldsymbol{A}^{l-k}\boldsymbol{B}, \quad \boldsymbol{G}_B(l) = \sum_{k=0}^{l} w_B(k)\boldsymbol{C}\boldsymbol{A}^{l-k}, \\
\boldsymbol{H}_{ij}(l) &= \sum_{k=0}^{l} \boldsymbol{A}^k\boldsymbol{b}_j\boldsymbol{c}_i\boldsymbol{A}^{l-k}, \quad \boldsymbol{H}_A(l)_{ij} = \sum_{k=0}^{l} w_A(k)\boldsymbol{H}_{ij}(l-k),
\end{aligned} \tag{12}$$

with $w_A(k)$, $w_B(k)$, and $w_C(k)$ denoting the unit-pulse responses of frequency-weighting functions $W_A(z)$, $W_B(z)$, and $W_C(z)$, respectively.

If a coordinate transformation is defined by:

$$\overline{\boldsymbol{x}}(k) = \boldsymbol{T}^{-1}\boldsymbol{x}(k) \tag{13}$$

and if it is applied to the linear system in (1), then we obtain a new realization $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{C}}, \boldsymbol{D})_n$ characterized by:

$$\begin{aligned} &\bar{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T},\ \bar{\boldsymbol{B}} = \boldsymbol{T}^{-1}\boldsymbol{B},\ \bar{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{T} \\ &\bar{\boldsymbol{W}}_B = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{W}_B\boldsymbol{T},\ \bar{\boldsymbol{K}}_c = \boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-\mathrm{T}} \end{aligned}. \tag{14}$$

From (2) and (14), it is clear that the transfer function $\boldsymbol{H}(z)$ is invariant under the coordinate transformation in (13). For the new realization, the frequency-weighted l_2- sensitivity measure in (8) is changed to

$$S(\boldsymbol{T}) = \sum_{i=1}^{p}\sum_{j=1}^{q} \mathrm{tr}\left[\boldsymbol{T}^{\mathrm{T}}\boldsymbol{M}_A(\boldsymbol{T})_{ij}\boldsymbol{T}\right] + q\,\mathrm{tr}\left[\bar{\boldsymbol{W}}_B\right] + p\,\mathrm{tr}\left[\bar{\boldsymbol{K}}_c\right], \tag{15}$$

where

$$\boldsymbol{M}_A(\boldsymbol{T})_{ij} = \sum_{l=0}^{\infty} \boldsymbol{H}_A^{\mathrm{T}}(l)_{ij}\boldsymbol{T}^{-\mathrm{T}}\boldsymbol{T}^{-1}\boldsymbol{H}_A(l)_{ij}. \tag{16}$$

If l_2-scaling constraints are imposed on the new state-variable vector $\bar{\boldsymbol{x}}(k)$, then it is required that:

$$(\bar{\boldsymbol{K}})_{ii} = (\boldsymbol{T}^{-1}\boldsymbol{K}\boldsymbol{T}^{-\mathrm{T}})_{ii} = 1 \ \text{ for } \ i = 1, 2, \cdots, n \tag{17}$$

where $\boldsymbol{K}$ is the controllability Gramian of the state-space model in (1), defined by:

$$\boldsymbol{K} = \frac{1}{2\pi j}\oint_{|z|=1} (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}(z^{-1}\boldsymbol{I}_n - \boldsymbol{A}^{T})^{-1}\frac{\mathrm{d}z}{z}, \tag{18}$$

which can be obtained by solving the Lyapunov equation:

$$\boldsymbol{K} = \boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}. \tag{19}$$

As a result, the minimization problem of a frequency weighted l_2- sensitivity measure subject to l_2- scaling constraints is now formulated as follows: *Given coefficient matrices* $\boldsymbol{A}$*,* $\boldsymbol{B}$ *and* $\boldsymbol{C}$*, obtain an* $n \times n$ *nonsingular matrix* $\boldsymbol{T}$ *which minimizes (15) subject to the* l_2*-scaling constraints in (17).*

3. **PROBLEM SOLUTION**

When the linear system in (1) is stable and controllable, the controllability Gramian $\boldsymbol{K}$ is symmetric and positive-definite [18]. This implies that $\boldsymbol{K}^{1/2}$ satisfying $\boldsymbol{K} = \boldsymbol{K}^{1/2}\boldsymbol{K}^{1/2}$ is also symmetric and positive-definite. Defining:

$$\hat{\boldsymbol{T}} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{K}^{-\frac{1}{2}}, \tag{20}$$

the l_2-scaling constraints in (17) can be expressed as

$$\left(\hat{\boldsymbol{T}}^{-\mathrm{T}}\hat{\boldsymbol{T}}^{-1}\right)_{ii} = 1 \quad \text{for} \quad i = 1, 2, \cdots n. \tag{21}$$

The constraints in (21) simply state that each column in $\hat{\boldsymbol{T}}^{-1}$ must be a unity vector. If matrix $\hat{\boldsymbol{T}}^{-1}$ is assumed to have the form:

$$\hat{\boldsymbol{T}}^{-1} = \left[\frac{\boldsymbol{t}_1}{\|\boldsymbol{t}_1\|}, \frac{\boldsymbol{t}_2}{\|\boldsymbol{t}_2\|}, \cdots, \frac{\boldsymbol{t}_n}{\|\boldsymbol{t}_n\|}\right], \tag{22}$$

then (21) is always satisfied. From (20) it follows that (15) is changed to

$$J_o(\hat{\boldsymbol{T}}) = \mathrm{tr}\left[\hat{\boldsymbol{T}} \sum_{i=1}^{p}\sum_{j=1}^{q} \hat{\boldsymbol{M}}_A(\hat{\boldsymbol{T}})_{ij} \hat{\boldsymbol{T}}^{\mathrm{T}}\right] + q\,\mathrm{tr}\left[\hat{\boldsymbol{T}}\hat{\boldsymbol{W}}_B\hat{\boldsymbol{T}}^{\mathrm{T}}\right] + p\,\mathrm{tr}\left[\hat{\boldsymbol{T}}^{-\mathrm{T}}\hat{\boldsymbol{K}}_c\hat{\boldsymbol{T}}^{-1}\right], \tag{23}$$

where

$$\begin{aligned} \hat{\boldsymbol{M}}_A(\hat{\boldsymbol{T}})_{ij} &= \sum_{l=0}^{\infty} \hat{\boldsymbol{H}}_A^{\mathrm{T}}(l)_{ij}\hat{\boldsymbol{T}}^{-1}\hat{\boldsymbol{T}}^{-\mathrm{T}}\hat{\boldsymbol{H}}_A(l)_{ij} \\ \hat{\boldsymbol{H}}_A(l)_{ij} &= \boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{H}_A(l)_{ij}\boldsymbol{K}^{\frac{1}{2}} \\ \hat{\boldsymbol{W}}_B &= \boldsymbol{K}^{\frac{1}{2}}\boldsymbol{W}_B\boldsymbol{K}^{\frac{1}{2}}, \quad \hat{\boldsymbol{K}}_c = \boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{K}^{-\frac{1}{2}} \end{aligned}. \tag{24}$$

From the foregoing arguments, the problem of obtaining an $n \times n$ nonsingular matrix $\boldsymbol{T}$ which minimizes (15) subject to the scaling constraints in (17) can be converted into an unconstrained optimization problem of obtaining an $n \times n$ nonsingular matrix $\hat{\boldsymbol{T}}$ which minimizes (23).

Now we apply a quasi-Newton algorithm [17] to minimize (23) with respect to matrix $\hat{\boldsymbol{T}}$ given by (22). Define:

$$\boldsymbol{x} = \left[\boldsymbol{t}_1^{\mathrm{T}}, \boldsymbol{t}_2^{\mathrm{T}}, \cdots, \boldsymbol{t}_n^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{25}$$

Then $J_o(\hat{\boldsymbol{T}})$ is a function of $\boldsymbol{x}$, denoted by $J(\boldsymbol{x})$. The algorithm starts with a trivial initial point $\boldsymbol{x}_0$ obtained from an initial assignment $\hat{\boldsymbol{T}} = \boldsymbol{I}_n$. Then, in the kth iteration a quasi-Newton algorithm updates the most recent point $\boldsymbol{x}_k$ to point $\boldsymbol{x}_{k+1}$ as [17].

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}_k, \tag{26}$$

where

$$
\begin{aligned}
\boldsymbol{d}_k &= -\boldsymbol{S}_k \nabla J(\boldsymbol{x}_k),\\
\alpha_k &= \arg\min_{\alpha} J(\mathbf{x}_k + \alpha_k \boldsymbol{d}_k),\\
\boldsymbol{S}_{k+1} &= \boldsymbol{S}_k + \left(1 + \frac{\boldsymbol{\gamma}_k^{\mathrm{T}} \boldsymbol{S}_k \boldsymbol{\gamma}_k}{\boldsymbol{\gamma}_k^{T} \boldsymbol{\delta}_k}\right) \frac{\boldsymbol{\delta}_k \boldsymbol{\delta}_k^{\mathrm{T}}}{\boldsymbol{\gamma}_k^{T} \boldsymbol{\delta}_k} - \frac{\boldsymbol{\delta}_k \boldsymbol{\gamma}_k^{\mathrm{T}} \boldsymbol{S}_k + \boldsymbol{S}_k \boldsymbol{\gamma}_k \boldsymbol{\delta}_k^{\mathrm{T}}}{\boldsymbol{\gamma}_k^{T} \boldsymbol{\delta}_k},\\
\boldsymbol{S}_0 &= \boldsymbol{I},\ \boldsymbol{\delta}_k = \boldsymbol{x}_{k+1} - \boldsymbol{x}_k,\ \boldsymbol{\gamma}_k = \nabla J(\boldsymbol{x}_{k+1}) - \nabla J(\boldsymbol{x}_k).
\end{aligned}
\tag{27}
$$

Here, $\nabla J(\boldsymbol{x})$ is the gradient of $J(\boldsymbol{x})$ with respect to $\boldsymbol{x}$, and $\boldsymbol{S}_k$ is a positive-definite approximation of the inverse Hessian matrix of $J(\boldsymbol{x})$. This iteration process continues until

$$
\left| J(\boldsymbol{x}_{k+1}) - J(\boldsymbol{x}_k) \right| < \varepsilon \tag{28}
$$

is satisfied where $\varepsilon > 0$ is a prescribed tolerance. If the iteration is terminated at step k, then $\boldsymbol{x}_k$ is viewed as a solution point.

The implementation of (26) requires the gradient of $J(\boldsymbol{x})$. Closed-form expressions for $\nabla J(\boldsymbol{x})$ are given below.

$$
\begin{aligned}
\frac{\partial J_o(\hat{\boldsymbol{T}})}{\partial t_{\xi\zeta}} &\cong \lim_{\Delta \to 0} \frac{J_o(\hat{\boldsymbol{T}}_{\xi\zeta}) - J_o(\hat{\boldsymbol{T}})}{\Delta},\\
&= 2(\beta_1 + \beta_2 + \beta_3 + \beta_4),
\end{aligned}
\tag{29}
$$

where $\hat{\mathbf{T}}_{\xi\zeta}$ is the matrix obtained from $\hat{\boldsymbol{T}}$ with its (ξ,ζ)th component perturbed by Δ [18, p. 655]

$$
\begin{aligned}
\hat{\boldsymbol{T}}_{\xi\zeta} &= \hat{\boldsymbol{T}} - \frac{\hat{\boldsymbol{T}} \boldsymbol{g}_{\xi\zeta} \boldsymbol{e}_\zeta^{\mathrm{T}} \hat{\boldsymbol{T}}}{1 + \boldsymbol{e}_\zeta^{\mathrm{T}} \hat{\boldsymbol{T}} \boldsymbol{g}_{\xi\zeta}},\quad \beta_1 = \boldsymbol{e}_\zeta^{\mathrm{T}} \hat{\boldsymbol{T}} \left[\sum_{i=1}^{p} \sum_{j=1}^{q} \hat{\boldsymbol{M}}_A(\hat{\boldsymbol{T}})_{ij} \right] \hat{\boldsymbol{T}}^{\mathrm{T}} \hat{\boldsymbol{T}} \boldsymbol{g}_{\xi\zeta},\\
\beta_2 &= \boldsymbol{e}_\zeta^{\mathrm{T}} \hat{\boldsymbol{T}}^{-\mathrm{T}} \left[\sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{l=0}^{\infty} \hat{\boldsymbol{H}}_A(l)_{ij} \hat{\boldsymbol{T}}^{\mathrm{T}} \hat{\boldsymbol{T}} \hat{\boldsymbol{H}}_A^{\mathrm{T}}(l)_{ij} \right] \boldsymbol{g}_{\xi\zeta},\\
\beta_3 &= q \boldsymbol{e}_\zeta^{\mathrm{T}} \hat{\boldsymbol{T}} \hat{\boldsymbol{W}}_B \hat{\boldsymbol{T}}^{\mathrm{T}} \hat{\boldsymbol{T}} \boldsymbol{g}_{\xi\zeta},\quad \beta_4 = p \mathrm{e}_\zeta^{\mathrm{T}} \hat{\boldsymbol{T}}^{-\mathrm{T}} \hat{\boldsymbol{K}}_c \boldsymbol{g}_{\xi\zeta},\\
\boldsymbol{g}_{\xi\zeta} &= -\partial \left\{ \frac{\boldsymbol{t}_\zeta}{\|\boldsymbol{t}_\zeta\|} \right\} \Big/ \partial t_{\xi\zeta} = \frac{1}{\|\boldsymbol{t}_\zeta\|^3} \left(t_{\xi\zeta} \boldsymbol{t}_\zeta - \|\boldsymbol{t}_\zeta\|^2 \boldsymbol{e}_\zeta \right),
\end{aligned}
\tag{30}
$$

where $\boldsymbol{e}_\zeta$ denotes an $n \times 1$ unit vector whose ξ^{th} element equals unity.

4. NUMERICAL EXAMPLE

Consider a two-input/three-output linear discrete-time system $(\boldsymbol{A}_o,\boldsymbol{B}_o,\boldsymbol{C}_o,\boldsymbol{D})_n$ specified by

$$\mathbf{A}_0=\begin{bmatrix}0&0&0.072&0&1.50\\1&0&0.300&0&0.20\\0&1&-0.100&0&0.90\\0&0&0&0&0.05\\0&0&0&1&0.40\end{bmatrix},\ \mathbf{B}_0=\begin{bmatrix}1&0\\0&0\\0&0\\0&1\\0&0\end{bmatrix}$$
$$\mathbf{C}_0=\begin{bmatrix}1.1&2.7&0.9&0.4&1.5\\2.1&3.1&0.3&0.2&0.1\\5.4&1.6&-1.7&-6.6&3.0\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}1.0&0.8\\0.3&0.6\\0.5&0.4\end{bmatrix}. \tag{31}$$

The frequency-weighting functions used in this example were given by a FIR digital low-pass filter with the following unit-sample response:

$$\begin{aligned}&\omega_A(i)=\omega_B(i)=\omega_C(i)=\\&\quad=0.256322\exp[-0.103203(i-4)^2]\end{aligned} \tag{32}$$

for $0\le i\le 20$, and zero elsewhere.

After carrying out the l_2- scaling for the above system with a diagonal coordinate transformation matrix, the frequency weighted l_2- sensitivity of the scaled system was computed from (8) as: S = 376.189355. Applying the quasi-Newton algorithm in (26) to the scaled realization, after 30 iterations, we obtained matrix $\hat{\boldsymbol{T}}$ as

$$\hat{\boldsymbol{T}}=\begin{bmatrix}0.961175&0.303147&-0.842605&-0.179240&0.226268\\-0.495649&1.088065&0.221605&0.132055&0.172347\\0.393141&-0.504370&1.186363&0.295758&-0.140807\\0.375790&0.152211&-0.983311&0.753533&-0.350233\\-0.379810&-0.595426&-0.071203&0.023294&0.932259\end{bmatrix} \tag{33}$$

or equivalently, matrix $\boldsymbol{T}$ was derived as

$$\boldsymbol{T}=\begin{bmatrix}0.804169&-0.173391&0.524033&0.115267&-0.3281681\\0.245230&1.000748&-0.066326&-0.089143&-0.5721654\\-0.456892&0.420661&1.034735&-0.805875&-0.2368884\\-0.163162&0.134432&0.311590&0.748686&0.0177808\\0.331605&0.196350&0.001687&-0.348711&0.7905921\end{bmatrix}. \tag{34}$$

Then, the frequency-weighted l_2-sensitivity measure in (23) become

$$S(\boldsymbol{P}_{30}) = 290.917872, \tag{35}$$

and the profiles of the frequency-weighted l_2 – sensitivity during the first 30 iterations of the algorithm are shown in Fig. 1.

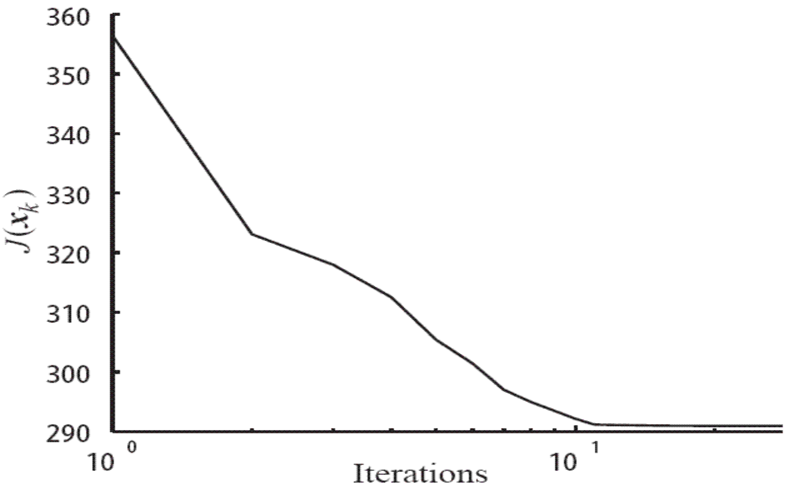

Fig. 1 – Profiles of frequency-weighted l_2 – sensitivity.

5. **CONCLUSIONS**

We have investigated the problem of minimizing the frequency-weighted l_2 – sensitivity measure subject to l_2 – scaling constraints for MIMO linear discrete-time systems. After converting the minimization problem of the frequency-weighted l_2 – sensitivity subject to l_2 – scaling constraints into an unconstrained optimization problem, an efficient quasi-Newton algorithm has been applied to solve the unconstrained optimization problem. The resulting coordinate transformation matrix has then been employed to construct the optimal MIMO system structure. Our simulation results have demonstrated the validity and effectiveness of the proposed technique.

It is noted that the sensitivity measure in [19] considers the sensitivity behavior of the transfer function at one frequency point to be as important as at another frequency point. In other words, a frequency-weighted sensitivity measure has not yet been considered in [19]. On the other hand, in this paper we consider a frequency-weighted sensitivity measure which corresponds to the more general case. Notice that solutions for frequency-weighted sensitivity minimization would be of practical use as these solutions allow to emphasize or de-emphasize the filter's sensitivity in certain frequency regions of interest.

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