A GEOMETRICAL METHOD FOR CONDUCTING SPHERES IN ELECTROSTATIC FIELD

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Key words: Electrical charge, Electrical images, Kelvin transform, Inversion method, Orthogonal spheres.

This paper treats the cases of a single and two orthogonal conductive spheres, situated in homogeneous electrostatic field. It starts from electrical image charge with respect to single sphere or orthogonal spheres. Determining the value and position of the image charges is done using geometrical methods based the Kelvin transform. The homogeneous electrostatic field is obtained from the displacement to infinity of the punctiform charge which is located outside the considered spheres.

1. INTRODUCTION

Kelvin transform use the geometrical inversion for space transformation. The rules regarding the transformation of charges and potentials from the inverse points are sufficient to compute the field and potential in case of punctiform charges, in the presence of conducting frames (spheres, planes, etc.). However there are some publications [1] that deal with the computation of electrical field and potential in the case of the single conducting sphere or two orthogonal spheres, in presence of external homogenous electrostatic field. This paper works with the electrical image charges in these situations obtained starting from the punctiform charge images. In comparison with other mathematical tools [2], the geometrical method presented in this work is more intuitive.

2. THE IMAGE OF A PUNCTIFORM CHARGE WITH RESPECT TO A CONDUCTIVE SPHERE WITH ZERO POTENTIAL

Suppose the case of a punctiform charge q situated at point M, in presence of a grounded conductive sphere of a radius a (Fig. 1). An arbitrary point P, located

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on the sphere, will be the pole of the inversion and the real number m^2 will be the power of the inversion.

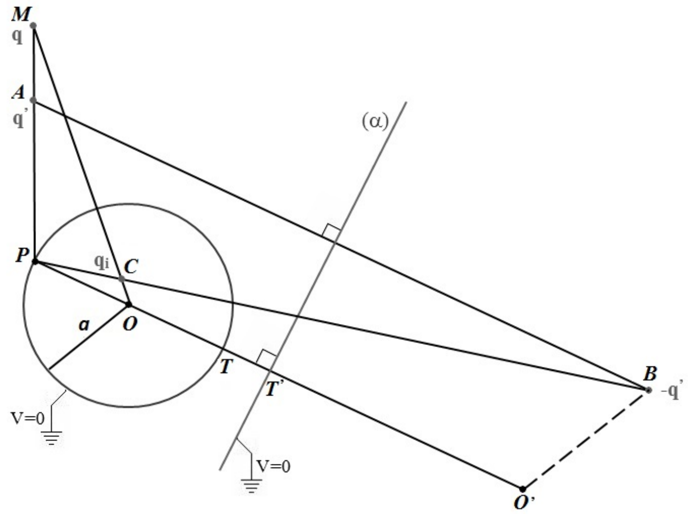

Fig. 1 – The considered punctiform charge and his corresponding image.

According to Fig. 1, by using Kelvin transform, through an original proof in [3], it is shown that the image charge q_i is placed in C:

$$OC = a^2 / OM, \tag{1}$$

with $q_i = -aq / OM$.

3. CHARGE IMAGES FOR THE CONDUCTING SPHERE LOCATED IN HOMOGENEOUS ELECTROSTATIC FIELD

Suppose the conducting sphere of radius a, free of electrical charge, placed in homogeneous electrostatic field $\vec{E}_0$ (Fig. 2). The classic computation of the problem is known. Starting from consideration that $\operatorname{rot}\vec{E} = 0$ or $\vec{E} = -\operatorname{grad} V$ and $\operatorname{div}\vec{E} = 0$ one reaches to Laplace equation in spherical coordinates $\nabla V(r, \theta) = 0$. Applying the method of separation of variables, integrating Euler and Legendre equations, imposing an unchanged $\vec{E}_0$ a large distance from the sphere and also imposing V_0 = const. at the sphere surface, leads to [4]:

$$V(r, \theta) = -E_0 r \cdot \cos\theta + \left(a^3 \cdot E_0 / r^2\right)\cos\theta + V_0, \tag{2}$$

$$\vec{E}(r,\theta)=\left(2a^3/r^3+1\right)E_0\cos\theta\cdot\vec{e}_r+\left(a^3/r^3-1\right)E_0\sin\theta\cdot\vec{e}_\theta\,. \tag{3}$$

If the sphere is grounded ($V_0=0$), the computing equation of field remains to be also the equation (3). The normal component of the flux density, at the surface of the sphere, gives the expression of the superficial charge $\rho_s=3\varepsilon_0E_0\cos\theta$ which appears due to electrostatic influence. It is important to note that the expression remains unchanged if the sphere is grounded or isolated, so the total charge on sphere is zero in both of cases.

Considering the sphere isolated from Fig. 2 and an electrical charge *q* placed in point *M*, at distance *D* from the centre of the sphere (Fig. 3).

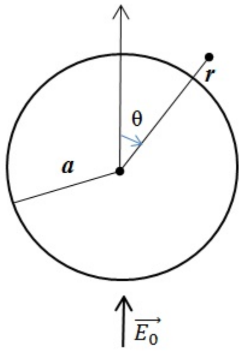

Fig. 2 – The conducting sphere.

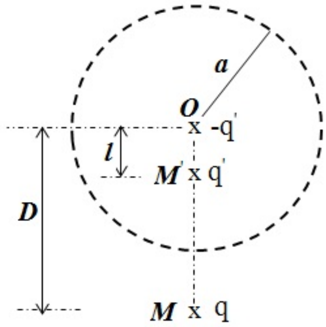

Fig. 3 – The electrical charge placed in point *M*.

According to results from the previous chapter, in point $M^{'}$ ($OM^{'}=a^2/D=l$) there is the charge $q^{'}=(-a/D)q$, so that *q* and $q^{'}$ ensure the potential zero on the assumed surface of the sphere. Being isolated, the total electrical charge must be zero, so that in *O*, an another image charge is considered: $-q^{'}=(a/D)q$. Therefore, the potential of the sphere is:

$$\left(a/D\right)\cdot q/4\pi\varepsilon_0a=q/4\pi\varepsilon_0D, \tag{4}$$

namely the potential imposed by the external charge, in the centre of the sphere.

In order to retrieve the situation presented in Fig. 2, suppose that charge *q*, located in point *M*, moves far away from the sphere, so that $D\to\infty$. In this situation, $l\to 0$ and the pair of charges ($-q^{'}$, $q^{'}$) form an electric dipole of a moment: $p=-q^{'}l=(aq/D)\cdot\left(a^2/D\right)=qa^3/D^2$ (to fit with definition of the dipole, admit that $q\to\infty$ like D^2). From the equalities (5) result that $p=4\pi\varepsilon_0a^3E_0$, E_0 being known.

$$E_0=\frac{q}{4\pi\varepsilon_0D^2}=\frac{p\left(D^2/a^3\right)}{4\pi\varepsilon_0D^2}=\frac{p}{4\pi\varepsilon_0a^3}\,. \tag{5}$$

In conclusion, in the case of homogeneous electrostatic field, the charge images form an electric dipole of a moment $\vec{p}$, where $\vec{p} = 4\pi\varepsilon_0 a^3 \vec{E}_0$, which is in respect with the field direction as highlighted in Fig. 4. In case of grounded sphere there is a single image charge so that the rules regarding the isolated sphere are not valid. In order to introduce the dipole the mathematical tricks from Fig. 5 are used. Two pairs of charges are introduced according to Fig. 5 which leads to zero potential at surface of the sphere.

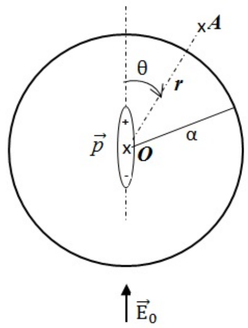

Fig. 4 – The electric dipole moment.

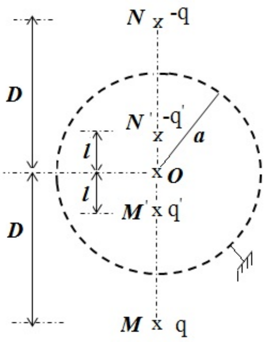

Fig. 5 – The applied matematical artifices.

For the dipole that corresponds to Fig. 5, the electric moment p is twice as large as in the case of the isolated sphere. The electric field is also twice as large in Fig. 5, so that $\vec{p} = 4\pi\varepsilon_0 a^3 \vec{E}_0$ remain valid.

4. CHARGE IMAGES FOR THE CONDUCTING SPHERE LOCATED IN HOMOGENEOUS ELECTROSTATIC FIELD

Taking into account the proof for the spheres of potential zero and extending the results with the necessary modifications for the other situations.

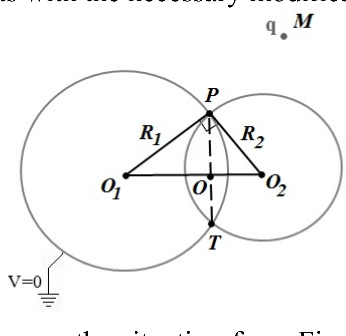

Fig. 6 – The punctiform charge and the orthogonal spheres.

Suppose the situation from Fig. 6 in which the given punctiform charge has the value q and is situated in the point M, outside the orthogonal grounded spheres with centres O_1 and O_2 and radii R_1 and R_2, respectively.The radical plane of the spheres intersects the plane of the drawing by line PT. Notice that PT is perpendicular to the centre line O_1O_2, their intersection being the point O. Because the spheres are orthogonal, the lines O_1P and O_2P are perpendicular.

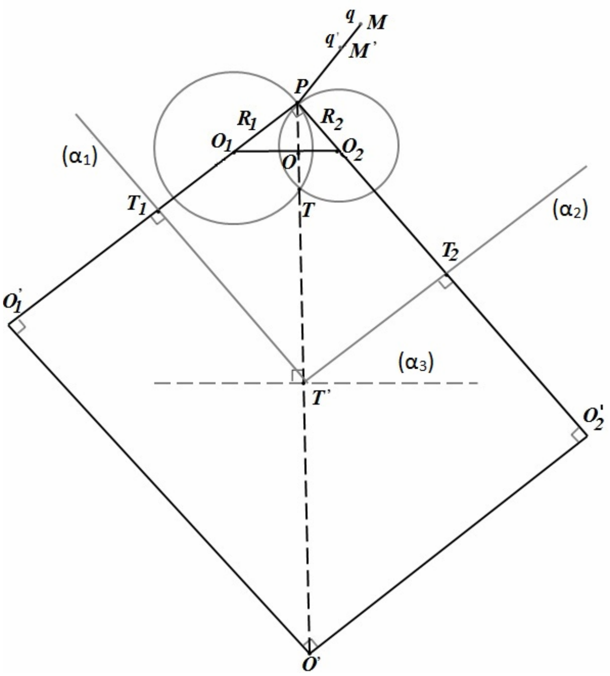

Fig. 7 – The first transform through inversion.

An inversion of pole P and power m^2 to the system consisting of the two spheres and the charge in point M, Fig. 7 is applied. The inverse of point M is $M^{'}$, as in Fig.7, so $PM \cdot PM^{'} = m^2$. According to Kelvin transform the charge from $M^{'}$ is $q^{'} = (m / PM)q$. The sphere centred in O_1 with radius R_1, noted $S(O_1, R_1)$, is transformed into the plane α_1 that is perpendicular to PO_1. The intersection of the plane and PO_1 is noted T_1 and is the inverse of the point diametrically opposite to P. The inverse of point O_1 is noted $O_1^{'}\left(PO_1^{'} = 2 \cdot PT_1\right)$ thus $O_1^{'}$ is the symmetric of point P with respect to the plane α_1. Similarly, the sphere centred in O_2 with radius R_2 (noted $S(O_2,R_2)$) is transformed into the plane α_2 that is perpendicular to PO_2. The intersection between the plane and PO_2 is noted T_2 and is the inverse of the point diametrically opposite to P.

The inverse of point O_2 is noted $O_2^{'}\left(PO_2^{'} = 2 \cdot PT_2\right)$ thus $O_2^{'}$ is the symmetric of point P with respect to the plane α_2. The intersection of the perpendicular planes α_1 and α_2 is line d that is perpendicular to the plane of the drawing (as in Fig. 7) in point T' where $PT \cdot PT' = m^2$. Line *d* is the inverse of the intersection circle of the spheres (a circle centred in *O* with diameter *PT*). Considering the virtual sphere centred in *O*, with diameter *PT*, the inverse of this sphere is the plane α_3 passing through *T'*, perpendicular to *PT*. The inverse of point *O* is *O'*, where $PO_2' = 2 \cdot PT_2$ so the rectangle $PT_1T'T_2$ is homothetic to the rectangle $PO_1'O'O_2'$. The center of this homothety is *P*, thus the point *T'* is the center of symmetry of the rectangle $PO_1'O'O_2'$.

The problem of the punctiform charge *q* in point *M* in the presence of orthogonal spheres with zero potential has transformed into a problem of the charge *q'* in point *M'* in the presence of perpendicular planes α_1 and α_2 of potential zero, a problem whose solution is known.

The image charges of the charge *q'* from *M'*, according to Fig. 8, are situated in the points M_1', M_2' and M_3'. M_1' and M_2' are the symmetric points of *M'* with respect to the planes α_1 and α_2, respectively and the value of their image charge is $-q'$. M_3' is the symmetric point of *M'* with respect to *T'*, the value of its image charge is *q'* [5].

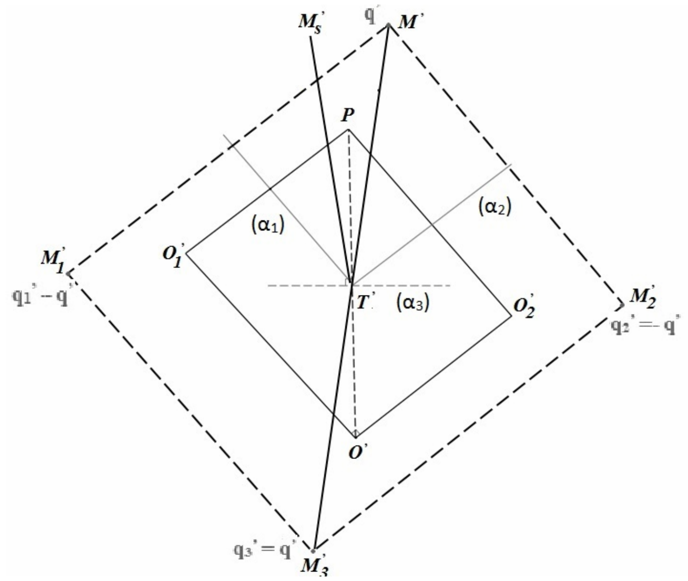

Fig. 8 – Image charge with respect to planes α_1, α_2, α_3.

Let M_s' be the symmetric point of *M'* with respect to the radical plane of the spheres (with respect to the line *PO'* from the plane of the drawing, as in Fig. 8). From the congruence of the triangles $\Delta O'T'M_3' = \Delta PT'M' = \Delta PT'M_s'$ the point M_s' is the symmetric point of M_3' with respect to the plane α_3.

By applying a second inversion with respect to the same pole P and power m^2, the charge q' from M' is transformed back into the given charge q from M. The point M_s' is transformed back into M_s, which is the symmetric point of M with respect to the radical plane of the spheres.

The planes α_1 and α_2, who have become fictive after introducing the image charges q_1', q_2', q_3' are transformed into the spheres $S(O_1, R_1)$ and $S(O_2, R_2)$. They become fictive after introducing the charges q_1, q_2, q_3 into which q_1', q_2', q_3' are transformed, after the final inversion that we are talking about.

It must be noticed that for each charge q_1', q_2', q_3' we can apply the result from the previous chapter, as though the charge were alone in the presence of the charge q from M. Thus M_1' is transformed into M_1 – the point where is the image charge of the charge in M with respect to the sphere $S(O_1,R_1)$, acording to Fig. 9, so $O_1M_1 = R_1^2 / O_1M$ and the value of the charge $q_1 = -(R_1 / O_1M)q$. The same for the point M_2' and the corresponding charge we is $O_2M_2 = R_2^2 / O_2M$ and the value of the charge $q_2 = -(R_2 / O_2M)q$. The charge q_3' from M_3' can be considered the image charge of the charge from M_s' with respect to the plane α_3. Because $q_3' = q' = (m / PM)q$ we have that the charge from M_s' is $-q_3' = -q' = -(m / PM)q$.

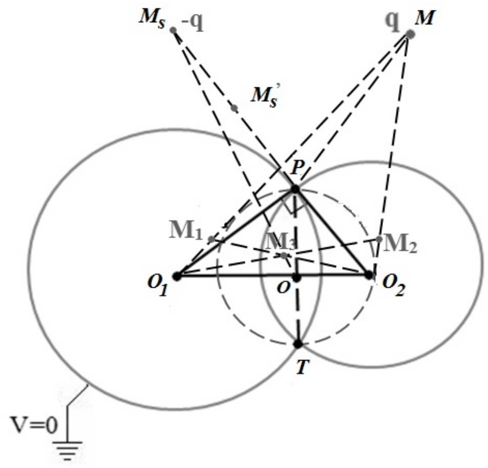

Fig. 9 – Positions M_1, M_2, M_3 of the image charges.

According to Chapter 2, the inverse of point M_3' will be M_3, meaning the point in which we put the image charge of the charge from M_s with respect to the virtual sphere of diameter PT. The radius of this sphere is $R = OP = R_1R_2 / \sqrt{R_1^2 + R_2^2}$, thus $OM_3 = R^2 / OM_s = R^2 / OM$. The value of the image charge, in M_3, is $q_3 = (R / OM)q$.

Remarks:

1. The points O_1, M_2, M_3 are collinear. Indeed, as P and $O_1^{'}$ are symmetric with respect to the plane α_1 and $M_2^{'}$ and $M_3^{'}$ are symmetric with respect to the plane α_1, the quadrilateral $PO_1^{'}M_3^{'}M_2^{'}$ is an isosceles trapezoid, thus inscriptible. Then the inverses O_1, M_2, M_3 of the points $O_1^{'}, M_2^{'}$ and $M_3^{'}$ are collinear.

2. In the same way the points O_2, M_1, M_3 are collinear. From 1) and 2) we have that M_3 is at the intersection of the lines O_1M_2 and O_2M_1.

3. The image charge of the charge q_2 from M_2 with respect to the sphere $S(O_1,R_1)$ is the charge q_3 from M_3.

For the proof one must show that: a) $O_1M_3 = R_1^2 / O_1M_2$ and b) $q_3 = -(R_1 / O_1M_2)q_2$.

Proof a). The points $O', M_3^{'}, M_2^{'}, O_2^{'}$ form an isosceles trapezoid, so are concyclic. Then their inverses are also concyclic, thus the quadrilateral $OM_3M_2O_2$ is inscriptible. The power of the point O_1 with respect to the circle passing through $OM_3M_2O_2$ is: $O_1M_3 \cdot O_1M_2 = O_1O \cdot O_1O_2$. But from the right triangle ΔO_1PO_2: $O_1O = R_1^2 / O_1O_2$. This, together with the preceding relation, yields to $O_1M_3 = R_1^2 / O_1M_2$.

Proof b). Taking into consideration the relations found previously for q_2 and q_3, $q_2 = -(R_2 / O_2M)q$ and $q_3 = -(R / OM)q$ the relation that must be proved $q_3 = -(R_1 / O_1M_2)q$ becomes (for $O_1O_2 \cdot R = R_1R_2$): $O_1O_2 \cdot OM = O_2M \cdot O_1M_2$ We write the double of the area of triangle ΔO_1O_2M in two ways:

$$O_1O_2 \cdot OM \sin(\angle MOO_2) = O_2M \cdot O_1M_2 \sin(\angle O_1M_2O_2). \tag{6}$$

But, $\angle MOO_2 = \angle M_sOO_1$ (because M and M_s are symmetric with respect to OP) and $\angle O_1M_2O_2 = \angle M_sOO_1$ (because the quadrilateral $OM_3M_2O_2$ is inscriptible, as we have shown before). Thus: $\sin(\angle MOO_2) = \sin(\angle O_1M_2O_2)$ so $O_1M_2 \cdot OM = O_2M \cdot O_2M_2$.

In the same way we show that the image charge of the charge q_1 from M_1 with respect to the sphere $S(O_2, R_2)$ is the charge q_3 from M_3.

We verify the fact that the potential [6] in the point P given by the charges q, q_1, q_2, q_3 from M, M_1, M_2, M_3 is zero:

$$\begin{aligned} V(P) &= \frac{1}{4\pi\varepsilon}\left(\frac{q}{MP} + \frac{q_1}{M_1P} + \frac{q_2}{M_2P} + \frac{q_3}{M_3P}\right) = \\ &= \frac{1}{4\pi\varepsilon}\left[\left(\frac{q}{MP} + \frac{q_1}{M_1P}\right) + \left(\frac{q}{MP} + \frac{q_2}{M_2P}\right) + \left(\frac{-q}{MP} + \frac{q_3}{M_3P}\right)\right] = 0. \end{aligned} \tag{7}$$

The result is obvious if we observe that each parenthesis is zero, representing the potential in a point on the sphere given by a charge exterior to the sphere, together with its image that was calculated in Chapter 2.

5. IMAGE CHARGES OF TWO ORTHOGONAL CONDUCTIVE SPHERES IN ELECTROSTATIC FIELD

Suppose orthogonal spheres from Fig. 10, in presence of homogeneous electrostatic field.

$$\vec{E} = E_0 \cdot \vec{e} = E_0\left(\cos\beta \cdot \vec{i} + \sin\beta \cdot \vec{j}\right). \tag{8}$$

The dipole images are introduced :

$$\vec{p}_1 = 4\pi\varepsilon_0 R_1^3 \vec{E}\,,\ \vec{p}_2 = 4\pi\varepsilon_0 R_2^3 \vec{E} \text{ and } \vec{p} = 4\pi\varepsilon_0 R^3 \vec{E}^{\,*}\,, \tag{9}$$

where:

$$\vec{E}^{\,*} = E_0\left(\cos\beta \cdot \vec{i} - \sin\beta \cdot \vec{j}\right). \tag{10}$$

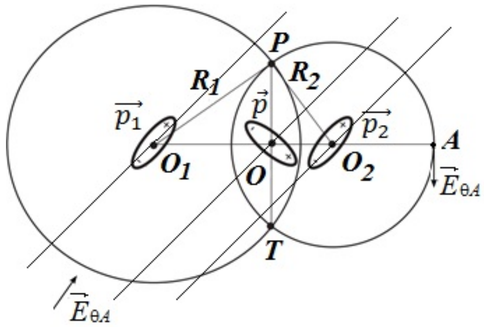

Fig. 10 – The orthogonal spheres, in presence of homogeneous electrostatic field.

The obtained equation are verified by determining the field tangential component in point A.

The obtained equations are verified by determining the field tangential component in point A. Using (9) the equation (11) is obtained. This was expected because the sphere surface R_2 is equipotential.

$$\vec{E}_{\theta A} = \left(\frac{p_1 \sin\beta}{4\pi\varepsilon_0 O_1A^3} + \frac{p_2 \sin\beta}{4\pi\varepsilon_0 O_2A^3} - \frac{p \sin\beta}{4\pi\varepsilon_0 OA^3} - E_0 \sin\theta\right)\vec{e}_\theta = 0\,. \tag{11}$$

6. CONCLUSIONS

Even though the results in Chapter 4 are well known, the purely geometrical proof used in the paper is original and much more intuitive than the vectorial one.

The determination of the dipolar charge images, in presence of homogeneous electrostatic field, is useful in many applications. The most well known case is that of the electromagnetic response of mixtures in which in a dielectric environment we have rare metallic inclusions [2]. The geometric method can be extended to other positions of the spheres when the angle between them is π/n [7–9].

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